Which wheel graphs are determined by their Laplacian spectra?

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\textbf{A B S T R A C T}

The wheel graph, denoted by \( W_{n+1} \), is the graph obtained from the circuit \( C_n \) with \( n \) vertices by adding a new vertex and joining it to every vertex of \( C_n \). In this paper, the wheel graph \( W_{n+1} \), except for \( W_7 \), is proved to be determined by its Laplacian spectrum, and a graph cospectral with the wheel graph \( W_7 \) is given.

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1. Introduction

Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G) \). All graphs considered here are simple and undirected. Let matrix \( A(G) \) be the \((0,1)\)-adjacency matrix of \( G \) and \( d_i \) the degree of the vertex \( v_i \). The matrix \( L(G) = D(G) - A(G) \) is called the \emph{Laplacian matrix} of \( G \), where \( D(G) \) is the \( n \times n \) diagonal matrix with \( \{d_1, d_2, \ldots, d_n\} \) as diagonal entries (and all other entries 0). The polynomial \( P_{L(G)}(\mu) = \det(\mu I - L(G)) \), where \( I \) is the identity matrix, is called the \emph{Laplacian characteristic polynomial} of \( G \), which can be written as \( P_{L(G)}(\mu) = q_0\mu^n + q_1\mu^{n-1} + \cdots + q_n \). Since the matrix \( L(G) \) is real and symmetric, its eigenvalues, i.e., all roots of \( P_{L(G)}(\mu) \), are real numbers, and are called the Laplacian eigenvalues of \( G \). Assume that \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (=0) \) are these eigenvalues; they compose the \emph{Laplacian spectrum} of \( G \).

Two non-isomorphic graphs are said to be cospectral with respect to the Laplacian spectrum if they share the same Laplacian spectrum \cite{1}. In the following, we call two graphs cospectral if they are cospectral with respect to the Laplacian spectrum.

Take two disjoint graphs \( G_1 \) and \( G_2 \). A graph \( G \) is called the \emph{disjoint union} (or sum) of \( G_1 \) and \( G_2 \), denoted as \( G = G_1 + G_2 \), if \( V(G) = V(G_1) \cup V(G_2) \) and \( E(G) = E(G_1) \cup E(G_2) \). Similarly, the \emph{product} \( G_1 \times G_2 \) denotes the graph obtained from \( G_1 \times G_2 \) by adding all the edges \((a, b)\) with \( a \in V(G_1) \) and \( b \in V(G_2) \). In particular, if \( G_2 \) consists of a single vertex \( b \), we write \( G_1 + b \) and \( G_1 \times b \) instead of \( G_1 + G_2 \) and \( G_1 \times G_2 \), respectively. In these cases, \( b \) is called an isolated vertex and a universal vertex, respectively. A \emph{subgraph} \cite{1} of graph \( G \) is constructed by taking a subset \( S \) of \( V(G) \) together with all vertices incident in \( G \) with some edge belonging to \( S \). Clearly, the product graph \( G_1 \times G_2 \) has a complete bipartite subgraph \( K_{m,n} \) where \( m \) and \( n \) are the order of \( G_1 \) and \( G_2 \), respectively.

Which graphs are determined by their spectra seems to be a difficult problem in the theory of graph spectra. Up to now, many graphs have been proved to be determined by their spectra \cite{2–8}. In \cite{3}, the so-called \emph{multi-fan graph} is constructed and proved to be determined by its Laplacian spectrum. Then, take the definition of the so-called \emph{multi-wheel graph}: The multi-wheel graph is the graph \((C_{n_1} + C_{n_2} + \cdots + C_{n_k}) \times b \), where \( C_{n_1} + C_{n_2} + \cdots + C_{n_k} \) is the disjoint union of circuits \( C_{n_i} \) and \( k \geq 1 \) and \( n_i \geq 3 \) for \( i = 1, 2, \ldots, k \). Note that the particular case of \( k = 1 \) in the definition is just the wheel graph.
$W_{n+1} = C_n \times b$ with $n+1$ vertices. In this paper, the wheel graph $W_{n+1}$, except for $W_7$, will be proved to be determined by its Laplacian spectrum. This method is also useful in proving that the multi-wheel graph $(C_n + C_{n_2} + \cdots + C_{n_k}) \times b$ is determined by its Laplacian spectrum, where $k \geq 2$. Here, we will skip the details of the proof for multi-wheel graphs. In [9], a new method (see Proposition 4 in [9]) is pointed out, which can be used to prove that every multi-wheel graph $(C_n + C_{n_2} + \cdots + C_{n_k}) \times b$ is determined by its Laplacian spectrum, where $k \geq 2$. But, for the wheel graph $W_{n+1}$, the new method in [9] is useless.

2. Preliminaries

Some previously established results about the spectrum are summarized in this section. They will play an important role throughout the paper.

Lemma 2.1 ([10]). Let $G_1$ and $G_2$ be graphs on disjoint sets of $r$ and $s$ vertices, respectively. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r (= 0)$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_s (= 0)$ are the Laplacian spectra of graphs $G_1$ and $G_2$, respectively, then $r + s; \mu_1 + s, \mu_2 + s, \ldots, \mu_{r-1} + s; \eta_1 + r, \eta_2 + r, \ldots, \eta_{s-1} + r$; and $0$ are the Laplacian spectra of graph $G_1 \times G_2$.

Lemma 2.2 ([11]).

(1) Let $G$ be a graph with $n$ vertices and $m$ edges and $d_1 \geq d_2 \geq \cdots \geq d_n$ its non-increasing degree sequence. Then some of the coefficients in $P_{L(G)}(\mu)$ are

$$q_0 = 1; \quad q_1 = -2m; \quad q_2 = 2m^2 - m - \frac{1}{2} \sum_{i=1}^{n} d_i^2;$$

$$q_{n-1} = (-1)^{n-1} n S(G); \quad q_n = 0$$

where $S(G)$ is the number of spanning trees in $G$.

(2) For the Laplacian matrix of a graph, the number of components is determined from its spectrum.

Lemma 2.3 ([12]). Let graph $G$ be a connected graph with $n \geq 3$ vertices. Then $d_2 \leq \mu_2$.

Lemma 2.4 ([13,11]). Let $G$ be a graph with $n \geq 2$ vertices. Then $d_1 + 1 \leq \mu_1 \leq d_1 + d_2$.

Lemma 2.5 ([14]). If $G$ is a simple graph with $n$ vertices, then $m_{C}(n) \leq \lfloor \frac{d_1}{n-d_1} \rfloor$, where $m_{C}(n)$ is the multiplicity of the eigenvalue $n$ of $L(G)$ and $\lfloor x \rfloor$ the greatest integer less than or equal to $x$.

Lemma 2.6 ([15]). Let $\overline{G}$ be the complement of a graph $G$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ be the Laplacian spectra of graphs $G$ and $\overline{G}$, respectively. Then $\mu_1 + \overline{\mu}_{n-i} = n$ for any $i \in \{1, 2, \ldots, n-1\}$.

Lemma 2.7 ([16]). Let $G$ be a connected graph on $n$ vertices. Then $n$ is an eigenvalue of Laplacian matrix $L(G)$ if and only if $G$ is the product of two graphs.

3. Main results

First, let us check that the graphs $G$ and $W_7$ in Fig. 1 are cospectral. By using Maple, the Laplacian characteristic polynomials of the graphs $G$ and $W_7$ are both

$$\mu^7 - 244\mu^6 + 231\mu^5 - 1140\mu^4 + 3036\mu^3 - 4128\mu^2 + 2240\mu.$$

That is, $G$ and $W_7$ are cospectral. Then, we will have the following proposition.

Proposition 3.1. The wheel graph $W_7$ is not determined by its Laplacian spectrum.

Theorem 3.2. The wheel graph $W_{n+1}$, except for $W_7$, is determined by its Laplacian spectrum.
Proof. Since the Laplacian spectrum of the circuit $C_n$ is $2 - 2 \cos \frac{2 \pi i}{n} (i = 1, 2, \ldots, n)$, by Lemma 2.1, the Laplacian spectrum of $W_{n+1}$ is $3 - 2 \cos \frac{2 \pi i}{n}$, where $i = 1, 2, \ldots, n - 1$, and also 0 and $n + 1$. Suppose that a graph $G$ is cospectral with $W_{n+1}$. Lemma 2.2 implies that graph $G$ has $n + 1$ vertices, $2n$ edges and one component. Then, by Lemma 2.7, $G$ is a product of two graphs. Let $d_1 \geq d_2 \geq \cdots \geq d_{n+1}$ be the non-increasing degree sequence of graphs $G$. By Lemma 2.3, $d_2 \leq \mu_2 \leq 5$, i.e., $d_2 \leq 5$. Lemma 2.4 implies that $d_1 + 1 \leq n + 1 \leq d_1 + d_2 \leq d_1 + 5$, i.e., $n - 4 \leq d_1 \leq n$. Consider the following cases for $d_1$.

Case 1. $d_1 = n - 4$. Since the multiplicity of the $\mu_1 = n + 1$ is 1, by Lemma 2.5, $1 \leq \frac{d_{n+1}}{n+1-(n-4)}$, i.e., $d_{n+1} \geq 5$. Then, $d_2 = d_3 = \cdots = d_n = d_{n+1} = 5$, i.e., there exist at least $n$ vertices of degree five in graph $G$. But, $5n + (n - 4) \neq 2 (2n)$, a contradiction to $\sum_{i=1}^{n+1} d_i = 2m$, where $m$ is the number of edges in $G$.

Case 2. $d_1 = n - 3$. Since the multiplicity of the $\mu_1 = n + 1$ is 1, by Lemma 2.5, $1 \leq \frac{d_{n+1}}{n+1-(n-3)}$, i.e., $d_{n+1} \geq 4$. Except for the vertex of degree $d_1 = n - 3$, suppose there still exist $x_5$ vertices of degree five and $x_4$ vertices of degree four in graph $G$. $\sum_{i=1}^{n+1} d_i = 2m$ implies the following equations:

\[
\begin{align*}
x_5 + x_4 + 1 &= n + 1 \\
x_5 + 4x_4 + (n - 3) &= 2 \times 2n.
\end{align*}
\]

Clearly, $x_5 = 3 - n$, $x_4 = 2n - 3$ is the solution of the equations. But $x_5 < 0$, a contradiction.

Case 3. $d_1 = n - 2$. By Lemma 2.5, $1 \leq \frac{d_{n+1}}{n+1-(n-2)}$, i.e., $d_{n+1} \geq 3$. Except for the vertex of degree $d_1 = n - 2$, suppose there still exist $x_5$ vertices of degree five, $x_4$ vertices of degree four and $x_3$ vertices of degree three in $G$. Lemma 2.2 and $\sum_{i=1}^{n+1} d_i = 2m$ imply the following equations:

\[
\begin{align*}
x_5 + x_4 + x_3 + 1 &= n + 1 \\
x_5 + 4x_4 + 3x_3 + (n - 2) &= 2 \times 2n \\
x_5 + 16x_4 + 9x_3 + (n - 2)^2 &= n^2 + 9n.
\end{align*}
\]

Clearly, $x_5 = 2n - 9$, $x_4 = 20 - 4n$, $x_3 = 3n - 11$. For $n = 4$, $x_5 < 0$, a contradiction. For $n = 5$, $x_5 = 1$, but $d_1 = 3 < 5$, a contradiction. For $n \geq 7$, $x_4 < 0$, a contradiction.

Case 4. $d_1 = n - 1$. By Lemma 2.5, $1 \leq \frac{d_{n+1}}{n+1-(n-1)}$, i.e., $d_{n+1} \geq 2$. Except for the vertex of degree $d_1 = n - 1$, suppose there still exist $x_5$ vertices of degree five, $x_4$ vertices of degree four, $x_3$ vertices of degree three and $x_2$ vertices of degree two in graph $G$. Lemma 2.2 and $\sum_{i=1}^{n+1} d_i = 2m$ imply the following equations:

\[
\begin{align*}
x_5 + x_4 + x_3 + x_2 + 1 &= n + 1 \\
x_5 + 4x_4 + 3x_3 + 2x_2 + (n - 1) &= 2 \times 2n \\
x_5 + 16x_4 + 9x_3 + 4x_2 + (n - 1)^2 &= n^2 + 9n.
\end{align*}
\]

By solving these equations, $x_4 = n - 3 - 3x_5$, $x_3 = 7 - n + 3x_5$, $x_2 = n - 4 - x_5$, where $x_5$ is an integer. And $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ imply that $\max(\frac{5n}{3}, 0) \leq x_5 \leq \min(\frac{5n}{3}, n - 4)$. Clearly, $\frac{5n}{3} < n - 4$ for $n \geq 5$. Therefore, if $n \geq 5$, then $x_3 > 0$, i.e., there must exist vertices of degree two in graph $G$. Note that $G$ is a product of two graphs and $G$ has a complete bipartite subgraph $K_{m_1, m_2}$, where $m_1 + m_2 = n + 1$. Then, for $n \geq 5$, the existence of vertices with degree two implies that the complete bipartite subgraph $K_{m_1, m_2}$ is $K_{n-1, 2}$ or $K_{n, 1}$. But for $K_{m_1, m_2} = K_{n, 1}$, there will exist a vertex with degree $n$ in graph $G$, a contradiction to $d_1 = n - 1$. For $n \geq 7$, $K_{n-1, 2}$ implies that there at least exist two vertices with degree no less than $n - 1$, a contradiction. Consider the following cases for $x_5$ and $n \leq 5$.

Case 4.1. $x_5 = 0$. Clearly, $x_4 = n - 3$, $x_3 = 7 - n$, $x_2 = n - 4$. Consider the following cases.

Case 4.1.1. $n = 3$. Clearly, $x_2 = -1 < 0$, a contradiction.

Case 4.1.2. $n = 4$. Clearly, $d_1 = 3$, $x_4 = 1$, $x_3 = 3$, $x_2 = 0$, but $d_1 = 3 < 4$, a contradiction.

Case 4.1.3. $n = 5$. Clearly, $d_1 = 4$, $x_4 = 2$, $x_3 = 2$, $x_2 = 1$, i.e., there exist three vertices of degree four, two vertices of degree three and one vertex of degree two in graph $G$. All the graphs with three vertices of degree four, two vertices of degree three and one vertex of degree two and with complete bipartite subgraph $K_{2, 4}$ have been enumerated; they are isomorphic to the graph shown in Fig. 2. By using Maple, the Laplacian characteristic polynomials of the graphs $G$ and $W_6$ are

\[
\begin{align*}
P_L(G)(\mu) &= \mu^6 - 20\mu^5 + 155\mu^4 - 580\mu^3 + 1044\mu^2 - 720\mu, \\
P_L(W_6)(\mu) &= \mu^6 - 20\mu^5 + 155\mu^4 - 580\mu^3 + 1045\mu^2 - 726\mu.
\end{align*}
\]

Clearly, they have different Laplacian characteristic polynomials, a contradiction.
Case 4.2. $x_5 \geq 1$. Clearly, for $3 \leq n \leq 5$, $x_4 = n - 3 - 3x_5 < 0$, a contradiction.

Case 5. If $d_1 = n$. Since both $G$ and $W_{n+1}$ have the largest degree $n$, $W_{n+1} = C_n + b$ and $G = C + b$, where $C_n$ is a unknown graph. Lemma 2.6 implies that $G$ and $W_{n+1}$ are cospectral, i.e., $C_n$ and $C$ are cospectral. Since the circuit $C_n$ is determined by its Laplacian spectrum [6], so is its complement $C_n$. Then, $G$ is isomorphic to $C_n$, i.e., $G$ is isomorphic to $W_{n+1}$. Therefore $G$ is isomorphic to $W_{n+1}$. □

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement [15], so the complements of all the wheel graphs $W_{n+1}$, except for $W_7$, are determined by their Laplacian spectra.

4. Conclusion

In this paper, the wheel graph $W_{n+1}$, except for $W_7$, is proved to be determined by its Laplacian spectrum by showing that a graph $G$ cospectral to the wheel graph $W_{n+1}$ must have a universal vertex, and this is the key point of the paper.

We would like to close this paper by posing an interesting question. Since the wheel graph $W_{n+1} = C_n \times b$ for $n \neq 6$ and the fan graph $F_{n+1} = P_n \times b$ (see [3]) are proved to be determined by their Laplacian spectrum, $C_n$ and $P_n$ are also determined by their Laplacian spectrum (see [6]); our question is that which graphs satisfy the following relation:

"If $G$ is a graph determined by its Laplacian spectrum, then $G \times b$ is also determined by its Laplacian spectrum."

If $G$ is disconnected, i.e., $G$ has at least two components, then the above relation is true (see Proposition 4 in [9]). But, if $G$ is connected, it is known that only the complete graph $K_n$, the circuit $C_n$ with $n \neq 6$ and the path $P_n$ satisfy the above relation until now.

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